Topological Semantics and Decidability

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Abstract

It is well-known that the basic modal logic of all topological spaces is $S4$. However, the structure of basic modal and hybrid logics of classes of spaces satisfying various separation axioms was until present unclear. We prove that modal logics of T_0, T_1 and T_2 topological spaces coincide and are $S4$. We also examine basic hybrid logics of these classes and prove their decidability; as part of this, we find out that the hybrid logics of T_1 and T_2 spaces coincide.

1 Basic definitions

In this paper we are going to study modal logics that arise as sets of all formulas valid on certain classes of topological spaces. Thus the first definition in this paper is bound to be about how the modal formulas are interpreted on topological spaces (topological semantics was first introduced by Tarski [3]).

Definition 1 (Topological semantics). A topological space is a pair (T, τ) where $\tau \subseteq \mathcal{P}(T)$ such that $\emptyset, T \in \tau$ and τ is closed under finite intersections and arbitrary unions. Elements of τ are called *opens*, an open containing a point x is called a *neighborhood* of the point x.

A *topological model* $\mathfrak{M}$ is a tuple (T, V) where $T = (T, \tau)$ is a topological space and the valuation $V : \textsc{Prop} \to \mathcal{P}(T)$ sends propositional letters to subsets of T.

Truth of a formula ϕ (of the basic modal language) at a point w in a topological model $\mathfrak{M}$ (denoted by $\mathfrak{M}, w \models \phi$) is defined inductively:

$$\begin{array}{lll}
\mathfrak{M}, w \models p & \text{iff} & x \in V(p) \\
\mathfrak{M}, w \models \phi \wedge \psi & \text{iff} & \mathfrak{M}, w \models \phi \text{ and } \mathfrak{M}, w \models \psi \\
\mathfrak{M}, w \models \neg\phi & \text{iff} & \mathfrak{M}, w \not\models \phi \\
\mathfrak{M}, w \models \Box\phi & \text{iff} & \exists O \in \tau \text{ such that } w \in O \text{ and } \forall v \in O.(\mathfrak{M}, v \models \phi)
\end{array}$$

The basic modal language can be extended with nominals and @ operator (in this case we call it $H(@)$) and universal modality A (we denote the dual modality E and call the language $H(E)$). Nominals are a special kind of propositional letters: it is required that their valuation is a singleton set. The semantics of @ and E is given below:

$$\begin{array}{lll} \mathfrak{M}, w \models @_i\varphi & \text{iff} & \exists v\, \mathfrak{M}, v \models i \text{ and } \mathfrak{M}, v \models \varphi \\ & & \text{(where } i \text{ is a nominal)} \\ \mathfrak{M}, w \models E\varphi & \text{iff} & \exists v\, \mathfrak{M}, v \models \varphi \end{array}$$

Relational and topological semantics are not completely unrelated; it is possible to transform certain topological spaces into frames and vice versa in a satisfiability-preserving fashion.

Proposition 1. *A topological space is called* Alexandroff *if every point of that space has a minimal neighborhood.*

For any Alexandroff space (T, τ) there exists a binary relation R such that for any valuation V and for any formula $\varphi \in H(E)$, $(T, R, V), w \models \varphi$ iff $(T, \tau, V), w \models \varphi$.

For any transitive reflexive frame (W, R) there exists a topology τ on W such that for any valuation V and for any formula $\varphi \in H(E)$, $(W, R, V), w \models \varphi$ iff $(W, \tau, V), w \models \varphi$.

Proof. For any point x of an Alexandroff space put Rxy for all $y \in O_x$ where O_x is the minimal neighborhood of O_x. The frame one obtains is a reflexive transitive frame.

For the second clause, the topology is defined as follows. Call a subset O of the frame $\mathfrak{F}$ *downward closed* if it follows from $x \in O$, Rxy that $y \in O$. The topology τ consists of all downward closed sets and is an Alexandroff topology.

In both cases it can be easily checked that the satisfiability of $H(E)$ formulas is preserved. $\square$

It is well-known that the (basic modal) logic of all topological spaces is $S4$. In what follows, we are going to deal with three classes of topological spaces, defined by the so-called separation axioms.

Definition 2 (Separation axioms). T_0 for any two distinct points x, y there is either an open neighborhood of x that does not contain y, or an open neighborhood of y that does not contain x.

T_1 any singleton set is closed

T_2 any two distinct points x, y can be separated by two open neighborhoods, i.e. there exist $O_x \ni x, O_y \ni y$ such that $O_x \cap O_y = \emptyset$.

There are necessary and sufficient conditions (given in [2]) of whether a class of spaces is definable in $H(@)$ (and $H(E)$). Thus, axioms T_0 and T_1 are definable in $H(@)$, the formulas are, respectively, $@_i\neg j \to (@_i\Box\neg j \vee @_j\Box\neg i)$ and $\Diamond i \to i$. On the other hand, [2] show that T_2 is not definable even in

$H(E)$. Basic modal language is even less expressible: none of the separation axioms is definable in it. The situation with separation axioms is a bit strange: although we know the boundaries of expressivity of modal and hybrid languages, we know very little about the structure of the logics. Are the logics of separation axioms distinct? Are they decidable? If yes, what is the complexity? In this paper we will address the first two questions, and we hope to use the techniques preseted here to get an answer for the third one.

2 Basic modal logic

In this section we will denote by $\mathsf{Log}(K)$ a set of formulas in basic modal language which are valid on all topological spaces in class K. We will prove that $Log(T_0) = Log(T_1) = Log(T_2) = S4$, our technique will be to build a topobisimulation between a finite topological space and a space from each respective class.

Definition 3 (Topobisimulation). Let (T, τ, V) and (S, σ, W) be two topological models and consider a relation $R \subseteq T \times S$. Denote

$$R(X) = \{y \mid \exists x \in X, (x, y) \in R\}$$
$$R^{-1}(Y) = \{x \mid \exists y \in Y, (x, y) \in R\}$$

for any subset $X \subseteq T$, $Y \subseteq S$.

The relation R is called a *topobisimulation* if

Prop if Rxy then for all $p \in \textsc{Prop}$, $(T, \tau), V, x \models p$ iff $(S, \sigma), W, y \models p$

Zig for any $O \in \tau$, $R(O) \in \sigma$

Zag for any $U \in \sigma$, $R^{-1}(U) \in \tau$

A bisimulation is called *total* iff for any $x \in T$ there is $y \in S$ such that Rxy and for any $y \in S$ there is $x \in T$ such that Rxy.

In topological semantics just like in the relational semantics, two points connected by a topobilimilation satisfy the same formulas (if the topobisimilation is total, this is true for the formulas with universal modality).

Definition 4 (Finite model property). A logic L has *finite model property* with respect to a class of topological models K if $K \models L$ and for any $\varphi \notin L$ there exists a finite $\mathfrak{M} \in K$ such that φ is satisfiable on $\mathfrak{M}$.

Proposition 2. *The logic $S4$ has a finite model property.*

Proof. The proof is Proposition 1 combined with the fact that $S4$ has a finite model property with respect to transitive reflexive frames. $\square$

Theorem 3. *The logic of T_0 spaces is $S4$.*

Proof. The inclusion $S4 \subseteq \mathsf{Log}(T_0)$ is obvious, so we only have to prove $\mathsf{Log}(T_0) \subseteq S4$. Take an arbitrary topological space (T, τ) and define an equivalence relation: $x \leftrightsquigarrow y$ iff for all $O \in \tau, x \in O$ iff $y \in O$. The quotient set with the maximal topology that makes the natural projection continuous (this topological space is known as Kolmogorov quotient of T) is a T_0 space. The graph of the natural projection map is a topobisimulation. It follows that every formula, that is not an $S4$ validity is not a T_0 validity either. $\square$

Theorem 4. *The logic of T_1 spaces is $S4$.*

Proof. By Proposition 2, $S4$ has finite (topological) model property, i.e. every formula that is not valid on the class of all topological spaces can be refuted on a finite topological model. We are going to build a topobisimulation between a finite topological model and a model based on a topological space with countable domain and T_1 topology. It will follow that any $S4$ non-theorem can be refuted on a T_1 space based model, hence $\mathsf{Log}(T_1) \subseteq S4$, hence $\mathsf{Log}(T_1) = S4$.

Let (T, τ, V) be a finite topological model, let us construct a topobisimilar model (S, σ, W) where (S, σ) is a T_1 topological space.

We will identify T with the initial segment of natural numbers, so $T = \{1, \dots, n\}$. First, let us introduce some notation:

$$X_k = \{nx + k \mid 0 \leq x < \infty\}, 1 \leq k \leq n$$

Let σ_0 be a cofinite topology on $\mathbb{N}$, that is

$$\sigma_0 = \{O \mid \mathbb{N} - O \text{ is finite}\}$$

and for any subset $O \subseteq T$ denote

$$\bar{O} = \{X_i \mid i \in O\}$$

Then define the topology σ on $S = \mathbb{N}$ to be generated by the set

$$\beta = \sigma_0 \cup \{\bar{O} \mid O \in \tau\}$$

β satisfies the finite intersection property, hence it indeed can serve as a base for topology. The topology generated by β consists of sets from σ_0 and sets of the form $\bar{O} - F$, where F is finite.

Define valuation to be

$$W(p) = \overline{V(p)} \text{ for all } p \in \textsc{Prop}$$

(S, σ) is a T_1 space, because σ contains σ_0, hence all complements of singleton sets are open.

Define a relation $R \subset T \times S$ as follows:

$$R = \{(k, l) \mid l \in X_k\}$$

Let us check that R is a topobisimulation. First, notice that for any $O \subseteq T$, $R(O) = \bar{O}$ and therefore is open by the definition of topology on R.

Second, notice that sets X_k are dense in (S, σ_0), as well as sets of the form $X_k - F$, where F is finite. It follows that any open set $U \in \sigma$ either has a non-empty intersection with every X_k, or is of the form $U = \bar{O} - F$ where F is finite. In the first case, $R^{-1}(U) = T$, in the second case $R^{-1}(U) = O$ and in both cases we get an open set.

It follows from the construction of valuations, that points connected by R agree on propositional letters. □

The main idea of the proof is that we could construct a space which is a disjoint union of dense subsets and then add necessary opens to the topology to get a space topobisimilar to T. We can exploit this idea in a more general setting, leading us to the following

Theorem 5. *The logic of T_2 spaces is $S4$.*

Proof. We will use essentially the same idea as in the proof of the Theorem 4: we will construct a bisimulation between a finite model and a T_2 model. In order to do this we will use a constriction by L. Feng and O. Masaveu. In the paper [1] they prove that for any cardinal α there exists a T_2 space which is a disjoint union of α dense subsets (such a space is called α-resolvable). We will apply this statement for a finite $\alpha = n$, so let (S, σ_0) be such a space and $S = \bigcup_{k=1}^{n} X_k$ where X_k are disjoint dense subsets from the theorem of Feng and Masaveu.

Now define $\bar{O}$, β, σ, the valuation W and the relation $R \subset T \times S$ the same way as in the proof of the previous theorem.

(S, σ) stays a T_2 space (because we have only added more opens to it). For any $O \subseteq T$, $R(O) = \bar{O}$ and is open by the definition of topology on R.

In order to prove **Zag** condition for R, we are going to prove that for any U which is a finite intersection of elements of β, $R^{-1}(U)$ is open. Since any open of σ is a union of sets of this form, **Zag** will follow.

Without loss of generality we can suppose that $U = I \cap \bar{O}$ for $I \in \sigma_0$ and some open $O \in \tau$. Since $I \in \sigma_0$ and $\bar{O}$ has a non-empty intersection with any set from σ_0, $R^{-1}(U) = R^{-1}(\bar{O}) = O$, which is open. **Prop** is immediate by construction of valuation. □

In fact, nothing in the proof depends on the T_2 condition, except the existence of n-resolvable sets. That leads us to the following more general result.

Theorem 6. *Let K be a class of topological spaces that contains an n-resolvable space for all finite n and for any space $(T,\tau) \in K$, it is true that $(T,\tau') \in K$ for all $\tau' \supset \tau$ (i.e. K is closed under refinement of topology). Then* $\mathsf{Log}(K) = S4$.

Proof. In this theorem we extract the key properties of the class T_2 used in the proof of the Theorem 5. Indeed, we need an n-resolvable space to start our construction, then we add new opens to this space in order to obtain a space bisimilar to the given finite topological space. If the class of topological spaces in question is closed under refinement of topology, we can do it the same way as we have done in the Theorem 5. $\square$

Remark 1. *Note that the proofs of Theorems 3, 4 and 5 still work for the basic modal language enriched with universal modality, because the bisimulations we construct are all total.*

3 Basic hybrid logic

In this section we will denote by $\mathsf{Log}(K)$ a set of formulas in the hybrid language $H(@)$ (with nominals and @) which are valid on all topological spaces in the class K.

In the subsequent subsections we will prove decidability of logics of different separation axioms. Our main tool will be the notion of topological filtration, which allows to present the information relevant for the satisfiability of a formula in a finite structure.

Definition 5 (Topological filtration). Let Σ be a subformula-closed set of formulas and $\mathfrak{M} = (T,\tau,V)$ be a topological model. Define an equivalence relation $\leftrightsquigarrow_\Sigma$ on T as follows:

$$w \leftrightsquigarrow_\Sigma v \text{ iff } \forall \varphi \in \Sigma\ \mathfrak{M}, w \models \varphi \text{ iff } \mathfrak{M}, v \models \varphi$$

A *filtration* of $\mathfrak{M}$ through Σ is a model $\mathfrak{N} = (S,\sigma,W)$, defined as follows. Let $S = T/\leftrightsquigarrow_\Sigma$ and let us denote by $[s]$ an equivalence class of $\leftrightsquigarrow_\Sigma$ with a representative s. For a formula $\varphi \in \Sigma$ define

$$[\![\varphi]\!]^{\mathfrak{N}} = \{[x] \mid M, x \models \varphi\}$$

and $W(p) = [\![p]\!]^{\mathfrak{N}}$. This is well-defined, because points from the same equivalence class satisfy the same formulas from Σ.

Let π be a natural projection map $t \mapsto [t]$. Define σ to be the finest topology that makes π continuous.

Note that if $\Sigma = Cl(\varphi)$ (all subformulas of a single formula φ), then any filtration by Σ is finite (there is only finite number of subsets of $Cl(\varphi)$).

3.1 T_1 spaces

The class T_1 does not have a finite model property: for example, the formula $i \to \Box i$ can only be falsified on an infinite model with T_1 topology. In order to prove decidability of $\mathsf{Log}(T_1)$ we will introduce a special kind of finite structures that represent infinite models — quasi-models. Quasi-models definition is designed in such a way that you can get a quasi-model out of any model by taking a filtration of it. At the same time any quasi-model can be obtained like this. That means that by exhausting the class of all possible quasi-models for a formula we exhaust the class of all possible models a formula can be satisfied on. Since there is a bound on a number of possible quasi-models for a formula, we get decidability.

Definition 6 (Hintikka set). Let Σ be a set of formulas closed under subformulas and single negations. A set $A \subseteq \Sigma$ is called a *Hintikka set* if it is maximal subset satisfying the following conditions:

1. $\bot \notin A$
2. if $\neg\varphi \in \Sigma$ then $\varphi \in A$ iff $\neg\varphi \notin A$
3. if $\varphi \wedge \psi \in \Sigma$ then $\varphi \wedge \psi \in A$ iff $\varphi \in A$ and $\psi \in A$

Definition 7 (Quasi-model). Let φ be a formula and $Cl(\varphi)$ be its subformula closure. A tuple (T, τ, λ), where (T, τ) is a finite topological space and λ is a function from T to $Cl(\varphi)$ is called a *quasi-model* for φ if the following holds:

1. $\lambda(t)$ is a Hintikka set for any $t \in T$
2. at least for one $t \in T$, $\varphi \in \lambda(t)$
3. if $O \in \tau$ and there is a formula $\Box\psi \in \Sigma$ such that $\forall t \in O\ \psi \in \lambda(t)$ then $\forall t \in O\ \Box\psi \in \lambda(t)$
4. if $i \in \lambda(t)$ where i is a nominal, then $T - \{t\} \in \tau$ (T_1 condition for quasi-models)

Here is a property of filtration that we will need for the main theorem

Lemma 7. *For any formula $\varphi \in \Sigma$, $[\![\Box\varphi]\!]^{\mathfrak{N}}$ is open.*

Proof. Suppose $[\![\Box\varphi]\!]^{\mathfrak{N}}$ is not open. Then let σ' be generated by $\sigma \cup [\![\Box\varphi]\!]^{\mathfrak{N}}$. Then we have for $O = \pi^{-1}([\![\Box\varphi]\!]^{\mathfrak{N}})$ that $\forall x \in O\ \mathfrak{M}, x \models \Box\varphi$ hence O is open which means that σ' makes π continuous. That contradicts the definition of σ. $\square$

Theorem 8. *A formula φ is satisfiable on a T_1 space iff there exists a topological quasi-model for it.*

Proof. The left-to-right direction can be proved using topological filtrations. Suppose we have a topological model $\mathfrak{M} = (T, \tau, V)$, where (T, τ) is a T_1-space and $\mathfrak{M}, v \models \varphi$, let $\Sigma = Cl(\varphi)$ and let $\mathfrak{N} = (S, \sigma, W)$ be the filtration of $\mathfrak{M}$ through Σ. Then define a quasi-model $Q = (S, \sigma, \lambda)$, where $\lambda([s]) = \{\psi \in Cl(\varphi) \mid \mathfrak{M}, s \models \psi\}$. It can be easily checked that $\lambda([x])$ is a Hintikka set for any $[x] \in T$. Next, $\varphi \in \lambda([v])$. The T_1 condition for quasi-models follows from Lemma 7. Let us finally show that the condition 3 holds. Suppose $O \in \sigma$, $\Box\psi \in \Sigma$ and $\forall s \in O\ \psi \in \lambda(s)$. Then $U = \pi^{-1}(O)$ is open and $\forall t \in U\ \mathfrak{M}, t \models \psi$, hence $\forall t \in U\ \mathfrak{M}, t \models \Box\psi$ and by construction of quasi-model $\forall s \in O\ \Box\psi \in \lambda(s)$.

To prove right-to-left direction we will construct a T_1 model $\mathfrak{M} = (S, \sigma, W)$ which satisfies φ from a given quasi-model (T, τ, λ). The model will have $\mathbb{N}$ as support. This is a reasonable choice, because any satisfiable hybrid formula is satisfiable on a countable or finite model, as we are going to prove. We can always allocate a finite open subspace to satisfy φ on by imposing a certain topology on $\mathbb{N}$, and in case φ can only be satisfied on an infinite model, we just have to add enough opens to cofinite topology on $\mathbb{N}$.

Just like in the previous section we identify T with the set of natural numbers $\{1, \dots, n\}$. Suppose there are m points $t_1, \dots, t_m \in T$ such that there is a nominal in $\lambda(t_k)$ for $1 \leq k \leq m$. If $m = n$, then every point is "named" by a nominal and should be represented by a singleton. In this case the model construction process described below will produce a model with a finite discrete submodel.

Denote $X_i = \{k\}$ for $i = t_k, 1 \leq k \leq m$ and let X_i for $i \in T - t_1, \dots, t_m$ form a partition of $\mathbb{N} - \{1, \dots, m\}$ such that every X_i is an infinite coinfinite set. Let σ_0 be a collection of cofinite subsets of $\mathbb{N}$ and for any subset $O \subseteq T$ denote

$$\bar{O} = \bigcup_{i \in O} X_i$$

Then define the topology on $\mathbb{N}$ to be generated by the following set:

$$\sigma = \sigma_0 \cup \{\bar{O} \mid O \in \tau\}$$

The valuation is defined the following way

$$W(p) = \bigcup_{p \in \lambda(k)} X_k \text{ for all } p \in \textsc{Prop} \cup \textsc{Nom}$$

The definition looks similar to the definition in Section 2 (indeed, the only real difference is in the definition of X_k); however, the proof works differently because of nominals.

For any formula ψ let us write $W(\psi)$ for $\{v \in S \mid \mathfrak{M}, v \models \psi\}$ and $V(\psi)$ for $\{v \in T \mid \psi \in \lambda(v)\}$. We will now prove the following

Claim 8.1. *For any $\psi \in Cl(\varphi)$, $W(\psi) = \overline{V(\psi)}$.*

Proof. The proof proceeds by induction on formula structure.

The statement for propositional letters and nominals follows from the construction of $\mathfrak{M}$. The Boolean connectives case as well as universal modality are easy to verify too.

Now, suppose that $\psi = \Box\chi \in Cl(\varphi)$ and the statement is true for χ. $W(\Box\chi) = \mathbb{I}W(\chi)$ is the largest open contained in $W(\chi)$. There are two possibilities here.

If $W(\chi)$ is cofinite, then it is open. Then $W(\Box\chi) = W(\chi) = \overline{V(\psi)}$. Note that by construction of topology σ, if $\bar{O}$ is open then O is open. And since $\overline{V(\psi)}$ is open (by condition 3 in the definition of a quasi-model), $\overline{V(\psi)} = \overline{V(\Box\psi)}$, which gives us the necessary statement.

If $W(\chi)$ is coinfinite (either finite, or infinite), then the only open sets that are contained in this set (if any) are of the form $\bar{O}-F$, where $O \in \tau$ and F is finite. So the largest open contained in $W(\chi)$ is of the form $\bar{O}$, where O is the largest open contained in $V(\chi)$, i.e. $O = \mathbb{I}V(\chi)$. In other words $W(\Box\chi) = \mathbb{I}W(\chi) = \overline{\mathbb{I}V(\chi)}$. By condition 3 in the definition of quasi-model $\mathbb{I}V(\chi) = V(\Box\chi)$. ☐

A direct consequence of this claim is that if $\varphi \in \lambda(k)$ then $\mathfrak{M}, v \models \varphi$ for all $v \in X_k$, which finishes the proof of the right-to-left direction. ☐

Note that the size of a quasi-model for φ is bounded by $2^{|Cl(\varphi)|}$ which allows us to deduce

Theorem 9. *$Log(T_1)$ is decidable.*

3.2 $\mathsf{Log}(T_1) = \mathsf{Log}(T_2)$

In the Theorem 5 we used the construction of the Theorem 4 and replaced the naturals with cofinite topology with an n-resolvable T_2 space whose existence is guaranteed by the theorem of Feng and Masaveu. In a similar fashion we are going to reuse the notion of the (T_1) quasi-model and modify the construction of the Theorem 8 in order to construct T_2 models out of the (T_1) quasi-models.

Theorem 10. *A formula φ is satisfiable on a T_2 space iff there exists a topological quasi-model for it.*

Proof. Since all T_2 spaces are T_1, the same filtration argument as in Theorem 8 applies here.

Now suppose we are given a quasi-model $\mathfrak{M} = (T, \tau, \lambda)$ for the formula φ. Let (S, σ_0) be an $n-m$-resolvable T_2 space, where $n = |T|$ and m is the number of points in the quasi-model, named by a nominal. Let $X'_1, \ldots, X'_{n-m}$ be the dense subsets of S which form the partition of S. Note that if $n > 1$

then these sets have empty interiors, because if one of them doesn't then no other can be dense. Let $X_{n-m+1}, \ldots, X_n$ be arbitrary singleton subsets of S. Finally, denote

$$X_i = X_i' - \bigcup_{j=n-m+1}^{n} X_j, \text{ for } 1 \leq i \leq 1, n-m$$

Since S is a T_1 space, $X_1, \ldots, X_{n-m}$ are still dense in S.

As usual, denote

$$\bar{O} = \bigcup_{i \in O} X_i$$

and consider a new topology σ on S generated by

$$\sigma_0 \cup \{\bar{O} \mid O \in \tau\}$$

and the valuation

$$W(p) = \bigcup_{p \in \lambda(k)} X_k \text{ for all } p \in \textsc{Prop} \cup \textsc{Nom}$$

It is left to prove that this construction preserves satisfiability of subformulas of φ. We use induction on formula structure just like in the proof of Claim 8.1. Boolean connectives and universal modality do not pose any problem, so we will only consider in detail the case when the formula is of the form $\Box\chi$.

We will use notation $V(\psi)$ and $W(\psi)$ in the same way as in the proof of Theorem 8. Suppose that $\psi = \Box\chi$ and $W(\chi) = \overline{V(\chi)}$.

First note that $X_1, \ldots, X_n$ have an empty interior in σ_0. This is obvious for $X_{n-m+1}, \ldots, X_n$ which are singleton sets, $X_1, \ldots, X_{n-m}$ have empty interiors since $X_1', \ldots, X_{n-m}'$ have empty interiors. This fact together with the induction hypothesis $W(\chi) = \bigcup_{i \in A} X_i$ implies that the only opens $W(\chi)$ contains are of the form $\bar{O} \cap F$ where $O \in \tau, O \subseteq A, F \in \sigma_0$. Thus information given by the topology of the quasi-model is enough to find $W(\Box\chi)$.

If $W(\chi)$ is cofinite then $W(\chi) = W(\Box\chi) = \overline{V(\Box\chi)}$, since $W(\chi)$ is open and condition 3 in the definition of a quasi-model holds. This is the only place in the present proof where we used the "T_1 condition" from the definition of the quasi-model; we have to require it because cofinite opens were already present in σ_0 and we inherited them in σ which is constructed after the quasi-model.

In case $W(\chi)$ is not cofinite,

$$\begin{aligned} W(\Box\chi) &= \mathbb{I}(\bigcup_{i\in A} X_i) = \\ &= \bigcup_{O\in\tau, O\subseteq A, F\in\sigma_0} (\bar{O}\cap F) = \\ &= \overline{\bigcup_{O\in\tau, O\subseteq A} \bar{O}} \\ &= \mathbb{I}(V(\Box\chi)) \end{aligned}$$

which finishes the proof.

□

Since every T_2 space is a T_1 space, we get the following corollary

Theorem 11. *The logic of T_2 spaces coincides with the logic of T_1 spaces (and hence, is decidable).*

3.3 T_0 spaces

We will next deal with the simplest and the most natural class of topological spaces, T_0 spaces.

Proposition 12. *An Alexandroff space corresponding to a partial order by the Proposition 1 is T_0 and the frame that corresponds to a T_0 Alexandroff space is a partial order.*

Proof. Let T be a T_0 Alexandroff space. Take two points $x, y \in T$ and suppose Rxy and Ryx. That means that x is in the minimal neighbourhood of x and that y is in the minimal neighborhood of x which contradicts the fact that T is a T_0 space. It follows that $\mathfrak{F}$ is antisymmetric, hence a partial order.

Now let $\mathfrak{F}$ be a partial order frame and let τ be the corresponding Alexandroff topology. Take two distinct points $x, y \in \mathfrak{F}$ and consider their corresponding pointwise generated subframes (that is, minimal neighborhoods) O_x and O_y. Since $\mathfrak{F}$ is antisymmetric, either $x \notin O_y$ or $y \notin O_y$, so τ is a T_0 topology. □

By the Proposition above, every T_0 validity is a partial order validity. The converse is not true.

Consider the countable topological space $(\mathbb{N}, \sigma)$ with cofinite topology. Construct a topological space (T, τ) as follows: let $T = \{*\} \cup \mathbb{N}$ and $\tau = \{U = \{*\} \cup O \mid O \in \sigma\}$. This is a T_0 space. Now, introduce a valuation that names $*$ with a nominal i and consider a formula $\varphi = \Diamond(\neg i \wedge \Diamond i)$. This formula is satisfied at $*$, but it is not satisfiable on any partial order. Hence $\mathsf{Log}(T_0)$ is a strict subset of the logic of partial order.

In order to understand the logic of T_0 spaces we will use again the technique of quasi-models. Although the counterexample we have just mentioned tells us that $Log(T_0)$ is more complicated than the logic of partial orders, it will serve us as the source of ideas on how one might build a T_0 model out of a quasi-model. We will need a different notion of a quasi-model than one for T_1 and T_2 spaces (otherwise $Log(T_0)$ would coincide with $Log(T_1)$ which is impossible).

The change to the definition will only affect one clause, namely the "T_1 condition for quasi-models". It is replaced with the following one:

> **(T_0 condition for quasi-models)** for every pair of points x, y named by nominals, there exists an open neighborhood O_x of x such that $y \notin O_x$ or there exists an open neighborhood O_y of y such that $x \notin O_y$.

Once again we will describe a way to construct a topological space (this time a T_1 space) that satisfies a given formula given a quasi-model for that formula. We will have as a consequence a

Theorem 13. *The* $\mathsf{Log}(T_0)$ *is decidable.*

Proof. What we really prove here is that T_0 satisfiability is equivalent to satisfiability an a quasi-model.

A filtration of a T_0 space through $Cl(\varphi)$ gives a T_0 quasi-model of φ, this can be verified directly, so we will not go into the details here.

The other direction of the proof goes as follows. Consider a T_0 quasi-model $\mathfrak{M} = (T, \tau, \lambda)$. We will identify T with natural numbers $1, \ldots, n$ and we will use such a numbering that $1, \ldots, m$ are named by a nominal, i.e. for $1 \leq km$, $i \in \lambda(k)$ for some nominal i. We will construct a topological model (S, σ, W) with a support $\{1, \ldots, m\} \cup \mathbb{N}$ and topology and valuation defined below. We will suppose further that $n \neq m$ since otherwise the quasi-model is a real model for φ already.

Partition S into sets $X_1, \ldots, X_n$: let $X_k = \{k\}$ for $1 \leq k \leq m$ and let $X_m, \ldots, X_n$ be the sets of the form $\{k + j(n-m) \mid 0 \leq j < \infty\}$ for $m \leq k \leq n\}$.

As usual, denote

$$\bar{O} = \bigcup_{i \in O} X_i$$

for $O \subseteq T$. Define the topology σ to be generated by the following collection of sets

$$\{\bar{O} - F \mid O \in \tau, F \subseteq \mathbb{N} \text{ finite } \}$$

Valuation is also defined in a usual way

$$W(p) = \bigcup_{p \in \lambda(k)} X_k \text{ for all } p \in \textsc{Prop} \cup \textsc{Nom}$$

Note that similarly to the Theorems 8 and 10, the only opens $\bar{A}$ contains (for $A \subset T$) are of the form $\bar{O} - F$ where $O \in \tau, O \subseteq A$, F is finite. This is what makes the same inductive proof as in the aforementioned theorems go through.

We will only mention that the model thus constructed is T_0. Any point x from $\mathbb{N}$ can be separated from any other point by a set $S - \{x\}$. Two points named by nominal can be separated by an open due to the T_0 condition for quasi-models.

□

Remark 2. *What happens if we add the universal modality to the language? The only part of proofs of Theorems 13, 8 and 10 that depends on the language is when we prove that a formula is satisfied at a point of a quasi-model iff it is satisfied at a corresponding point of the model we have constructed. This is proved by induction on formula structure, and it is easy to see that if we add a clause that handles universal modality, the statement will still hold. Thus, the aforementined theorems also apply to* $H(E)$.

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